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A METHOD OF DESCRIBING MISS DISTANCES FOR LUNAR AND INTERPLANETARY TRAJECTORIES

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pp. ii, 1-19

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21 pages

CONTENTS

FIGURES

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A METHOD OF DESCRIBING MISS DISTANCES FOR LUNAR AND INTERPLANETARY TRAJECTORIES*

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ABSTRACT

Miss distances for lunar and interplanetary trajectories can be described by specifying two components of the impact parameter, which is treated as a vector. This is analogous to the use of range and azimuth (or cross range) in specifying the impact point for terrestrial targets. The resulting coordinates are very nearly linear functions of the variables of the trajectory near the earth, except in cases where the trajectory is of the minimum-energy type, such as a Hohmann orbit. Applications are given to the theory of guidance and to a method for automatically searching for a trajectory which hits or misses the target in a specified manner.

*This paper presents the results of one phase of research carried out at the Jet Propulsion Laboratory, California Institute of Technology, under Contract No. NASw-6, sponsored by the National Aeronautics and Space Administration.

I. INTRODUCTION

In treating the errors of a vehicle trajectory which is intended to hit a target, such as Venus, or a point on the surface of the earth, it is convenient to construct a system of differential corrections. The differential corrections are partial derivatives of some measure of the miss distance with respect to the coordinates at the burnout or injection point, and they are particularly important in the theory and design of guidance systems.

If the target is a point on the surface of the earth, a satisfactory measure of miss distance is range and azimuth error. A typical differential correction, then, would be the partial derivative of range error with respect to speed at burnout.

For a "massy" point in space, however, some other measure of miss distance must be used. The seemingly obvious thing to do is to use the miss distance itself, i.e., the distance of closest approach of the vehicle to the center of the target. However if differential corrections are to be used for guidance or trajectory design purposes, then the use of this miss distance has certain drawbacks. In particular, for trajectories which pass close to the center of the target, the miss distance is a quadratic function of the initial conditions. Thus the variation of miss distance versus any of the initial coordinates is zero for a direct hitting trajectory. Another drawback in using the miss distance is that the direction of the path near the target is not given. For a terrestrial target this corresponds to an uncertainty in how much of the miss is due to an error in range, and how much to an error in azimuth.

A solution to the problem of specifying the miss distance is obtained by making use of the parameters of the hyperbola near the target. Components of the impact parameter, defined as in scattering theory in atomic physics but treated as a vector, are used to provide two quantities which eliminate the nonlinearities caused by the gravitational field of the target, and specify the direction of the miss.

The general problem of how many independent variables are observable near the target is investigated. This leads to a discussion of guidance equations.

II. A MEASURE OF MISS DISTANCE

The specification of miss distance that is used is based on the impact parameter, $\vec{B}$, which is computed from the osculating conic (see Appendix). The elements of the conic are computed when a fixed distance from the target is reached, the distance being determined by the point where a conic approximation for the remainder of the trajectory is sufficiently accurate, or when closest approach is attained, whichever occurs first. This is further discussed in the Appendix. $\vec{B}$, the position vector in the plane of the trajectory originating at the center of gravity of the target and directed perpendicularly to the incoming asymptote of the hyperbola (see Fig. 1), is approximately the vector miss which would occur if the target had no mass.

In constructing a system of differential corrections, all three components of $\vec{B}$ cannot be used because they are not independent. Since the asymptote of the hyperbola is approximately fixed in

direction (for interplanetary trajectories it varies less than 0.01 deg for expected changes in initial conditions), the components of $\vec{B}$ must satisfy the relation that $\vec{B}$ is orthogonal to the asymptote.

To account for this, two components of $\vec{B}$, which are analogous to the two quantities used to describe the miss for a terrestrial target, are used. A set of orthogonal unit vectors is computed. Let $\vec{S}$ be a unit vector in the direction of the incoming asymptote, $\vec{T}$ be a unit vector perpendicular to $\vec{S}$ that lies in a fixed plane such as the equatorial plane, and $\vec{R}$ be a unit vector to form a right-handed system.

$$\vec{R} = \vec{S} \times \vec{T} \tag{1}$$

Since $\vec{B}$ is perpendicular to $\vec{S}$, it lies in the plane determined by $\vec{R}$ and $\vec{T}$. The variables which are used are the projections of $\vec{B}$ on $\vec{T}$ and on $\vec{R}$, i.e., $\vec{B}\cdot\vec{T}$ and $\vec{B}\cdot\vec{R}$. The complete set of formulas for computing the vectors is given in the Appendix.

The relation between the miss distance and $\vec{B}$ will now be shown. If a particular value of the distance of closest approach is desired, the magnitude of $\vec{B}$ can be computed from the following relations concerning the parameters of the hyperbola. In rectangular coordinates

$$\frac{(x - c)^2}{a^2} - \frac{y^2}{b^2} = 1 \tag{2}$$

where a is the semimajor axis,

$$c = \sqrt{a^2 + b^2} \tag{3}$$

$$b = \left|\vec{B}\right| \tag{4}$$

a is essentially determined by the launch time and the time of flight, hence it can be determined from a trajectory which does not have the desired miss distance. Since the miss distance is c - a, b can be determined. For interplanetary trajectories, a can be neglected in comparison to b for large miss distances, such as fifty target diameter so that the miss distance is then approximately equal to b. By the components of $\vec{B}$, or $\vec{B}\cdot\vec{T}$ and $\vec{B}\cdot\vec{R}$, the direction of the miss is determine

These relations are shown in Fig. 2. The line OP represents the path directed at the center of the target. This line also represents the asymptote of the osculating conic, which is very nearly parallel to the asymptotes of other conics which intersect the surface of the target. Consider now the cases where the impact is not vertical and the impact parameter is along the $\vec{T}$ axis. Since the plane of the trajectory contains the original $\vec{S}$ vector the locus of the impact points on the surface of the target must be a great circle. This is true whenever the direction of $\vec{B}$ is held fixed and the magnitude is varied. It can also be seen that the angle between such great circles at P is the same as the angle between the corresponding $\vec{B}$ vectors.

The orientation of the asymptote has been investigated for many different kinds of trajectories. The greatest variations in the orientation occur in lunar trajectories, but even here the direction is fixed to within a fraction of a degree for expected variations in initial conditions.

III. APPLICATIONS TO HOMING

In searching for trajectories which miss the target in a specified manner one often has approximate values of the parameters that determine the powered flight, or the values of the coordinates at the end of the powered flight. Better estimates of the initial conditions are needed to insure a specified miss distance. The method used in accomplishing this is called homing.

The homing method proceeds by computing a reference trajectory based on the original estimate of the initial conditions, and then computing two trajectories where each one of two initial coordinates or parameters is varied while the other is kept fixed. By observing the resulting changes in $\vec{B}\cdot\vec{T}$ and $\vec{B}\cdot\vec{R}$ due to small changes in the input conditions, new imput conditions can be obtained by a linear interpolation scheme. If nonlinearities are suspected, then the size of the increments to the initial variables can be monitored so that they do not exceed values for which the linear interpolation method succeeds. Table 1 shows the number of iterations required by the homing method for two Venus trajectories. The time of launch and the firing angle were varied here.

Table 1. Example of the Homing Method for Two Venus Trajectories

Parameter		Trajectory A	Trajectory B
Time of Flight, days		107	96
Perihelion Distance, km		1.02×10^8	9.5×10^7
Distance to Venus from the Sun, km		1.08×10^8	1.08×10^8
Miss Distance, km	Trial 1	3.1×10^6	1.6×10^6
	Trial 2	9.6×10^5	5.0×10^4
	Trial 3	2.8×10^5	3.2
	Trial 4	1.4×10^3	
	Trial 5	0.13	

In the Table the perihelion distance is the minimum distance to the sun of the transfer ellipse. It can be seen that in Trajectory B, where the perihelion distance is smaller, the number of trials required is less. Whenever the perihelion distance of the transfer conic is nearly equal to the distance of Venus from the sun then the differential corrections become nonlinear and more iterations are required in homing.

The kind of lunar trajectories for which the homing method has been used is not the minimum-energy type. Generally only one or two iterations have been required to achieve the correct miss distance.

An interesting case occurred in homing on a Mars trajectory when a mistake was made in the input conditions and the right ascension was in error by 180 deg. Nevertheless, the homing method succeeded in varying the initial conditions such that a satisfactory miss distance could be attained.

IV. DEGREES OF FREEDOM

In order to generalize the ideas about miss distance and to introduce problems in guidance the following question will be treated: Given a launching location on the earth's surface, a vehicle configuration, and a range of firing times (limited by the rotation of the earth), what is the minimum number of quantities that can be specified concerning the trajectory near the target in order that the trajectory shall be completely determined? The answer to this question is connected with the formation of guidance equations, which must ensure that all or some of these quantities near the target attain prescribed values.

If the trajectory were not constrained to originate from a certain region at the earth and start during a given time interval, there would be six quantities that could be specified about the trajectory near the target corresponding to the six degrees of freedom of the vehicle. With the constraints imposed by the launching conditions there are only three degrees of freedom. This was discovered by examining many lunar trajectories and finding the subspace which contains the points of the allowable trajectories.

The reason for the loss of half of the degrees of freedom can be found by considering an interplanetary trajectory, such as one which approaches Mars. Instead of considering the actual trajectory, consider an equivalent trajectory where the path in the sun's region of influence is the same, but where the path near the earth is replaced by a fictitious path where the vehicle has an initial velocity near the earth and the earth is assumed to have no effect on the vehicle.

In other words the path near the earth is approximated by the asymptote to the original escape hyperbola. The family of asymptotes which represent the class of trajectories approaching Mars then seems to originate from a small region, within an order of magnitude of the size of Earth. Since this size is insignificant compared to the distances involved, the equivalent trajectories may be thought of as originating from a point source. There are three velocity components that can be varied. This corresponds to the three degrees of freedom near the target.

The essential part of the proof is that the trajectories seem to originate from a point source. A simplified version of the problem may be considered in which all gravitational fields are neglected. Trajectories then are straight lines which originate in one region in space and are directed towards a target point. It can easily be proved that as the distance between the target and the region of origin is increased, only three numbers are necessary to specify the trajectory near the target.

Two coordinates, $\vec{B} \cdot \vec{T}$ and $\vec{B} \cdot \vec{R}$ have been described; it remains to choose a third coordinate. The time of flight has been used as the third coordinate since this is often of direct interest. For a mission such as a soft landing on the moon the vis viva[1] or total energy near the target might be chosen, since the velocity at the surface of the target is then readily available. The vis viva and also the semimajor axis a are more linear functions than the time of flight, and therefore either one is an excellent choice for a third variable.

[1]Moulton, F. R., An Introduction to Celestial Mechanics (2nd Edition). The Macmilian Company, New York, 1914.

If this particular coordinate system is linear with respect to initial coordinate changes for a given mission, then since the system is complete (or has the maximum number of dimensions) any other system of coordinates which is linear must be a linear transformation of this system. Thus for missions which possess the linearity property, the given coordinate system can be used as a basis for any other system which is more convenient.

V. GUIDANCE EQUATIONS

Having discussed the meaning of the set of coordinates, their differential corrections will now be discussed. It will be shown that differential corrections can lead to guidance equations. In particular the use of a set of coordinates which is fairly linear provides some insight as to what form the guidance equations may take.

A method of analyzing guidance equations is to think of the equations as predicting the coordinates near the target. Let Q_i ($i = 1, 2,$ and 3) be the coordinates near the target, and q_j ($j = 1, 2, \ldots 6$) be the coordinates in the region where the vehicle is being guided. For simplicity assume that only linear terms need be used. Then the guidance equations are

$$\delta Q_i = \sum_{j=1}^{6} \frac{\partial Q_i}{\partial q_j} \delta q_j \quad i = 1, 2, 3 \tag{5}$$

The variations δq_j refer to instantaneous variations from the standard value. By steering the vehicle or by shutting the motor off at a nonstandard time the δQ_i can be nulled. It should be noted that the guidance equations incorporate the differential corrections.

Not all of these equations need be used. For instance if the time of flight need not be controlled then one equation is eliminated. If the standard point of closest approach to the target is far removed from the center of the target it may be feasible to eliminate another constraint and to guide along the direction from the standard miss distance to the center of the target and to disregard errors in the perpendicular direction. A useful set of variables to use to describe errors in the guided portion of the trajectory are:

R = the distance from the center of the earth (or other body)

V = speed

θ = angle between the standard perigee direction and the radius vector to the vehicle

Z = cross range (which is zero on the standard path)

$\dot{Z}$ = cross range rate

Γ = path angle measured with respect to the local horizontal

In this set of variables a change in θ is equivalent to a change in the ground range. The effects of changes in these variables for lunar trajectories are illustrated in Fig. 3. Here the values of the coordinate deviations for the initial coordinates were chosen so that $|\vec{B}|$ is equal to a thousand kilometers. The trajectories chosen all have the angle between the orbital plane near the earth inclined by an angle of 30 deg to the plane of the moon, and have an angle of 10 deg between the point of injection and perigee. Three different values of the <u>vis viva</u> near the earth, C_{3E}, were used.

It can be seen that in the three cases shown, and in practically all the cases that have been computed which represent all reasonable

lunar trajectories, that the coordinates R and V, θ and Γ, Z and $\dot{Z}$ can be paired insofar as the effects of the miss near the moon are concerned. For example an error in ground range θ can be compensated for by a change in path angle Γ. Another feature which emerges is that the effect of a change in Z is opposite to that produced by a change in $\dot{Z}$. Hence a scheme to cancel the effects of one by introducing a change in the other is unstable from the point of view of incurring large deviations from the standard flight path.

In Fig. 4 a similar plot is shown for a Mars trajectory. It shows a similarity to a fast lunar trajectory.

VI. CONCLUSIONS

It has been shown that the variables $\vec{B}\cdot\vec{T}$ and $\vec{B}\cdot\vec{R}$ can serve as a measure of miss distance which removes the nonlinearity caused by the attraction of the target. The variables are a function of the osculating conic near the target and as such can be exactly computed for any lunar or interplanetary trajectory. With the addition of a third variable, such as the time of flight, the set of variables near the target is complete. The set of variables can be used in much the same manner as range and azimuth error in analyzing terrestrial trajectories.

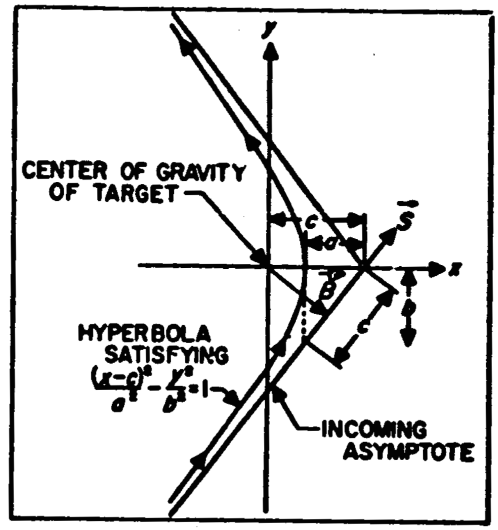


Fig. 1. Geometry Near Target

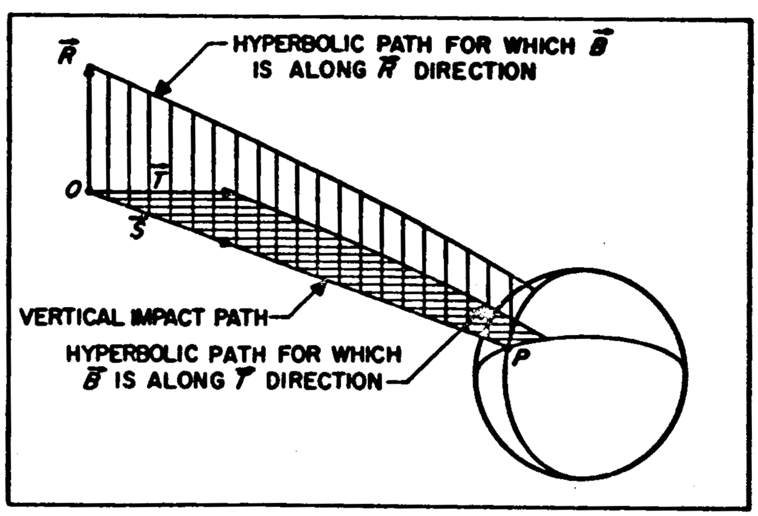
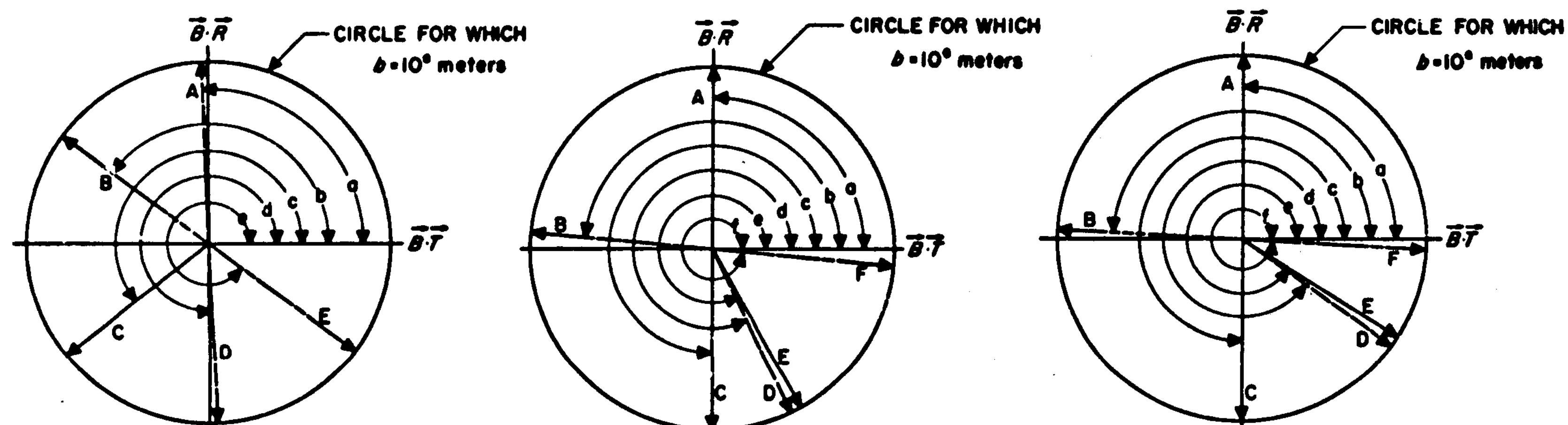


Fig. 2. Location of Impact Points on the Surface of the Target



TIME OF FLIGHT 79.5 hr $C_{3E} = -1.8 \times 10^6 \left(\frac{\text{meters}}{\text{sec}}\right)^2$

CASE A. $\Delta Z = 0.185017 \times 10^6$ meters
a. 323.9 deg
CASE B. $\Delta\theta = 0.191702$ deg.
b. 2. 272 deg
CASE C. $\begin{cases} \Delta R = 0.718876 \times 10^6 \text{ meters} \\ \Delta V = 0.601450 \text{ meters/sec} \end{cases}$
c. 218 deg
CASE D. $\Delta \dot{Z} = 114.103$ meters/sec
d. 143.5 deg
CASE E. $\Delta\Gamma = 0.949780 \times 10^{-1}$ deg
e. 91.2 deg

TIME OF FLIGHT 42.4 hr $C_{3E} = 1.4 \times 10^6 \left(\frac{\text{meters}}{\text{sec}}\right)^2$

CASE A. $\Delta Z = 0.189986 \times 10^6$ meters
a. 90 deg
CASE B. $\Delta\theta = 0.155277$ deg
b. 174.5 deg
CASE C. $\Delta \dot{Z} = 59.1199$ meters/sec
c. 270 deg
CASE D. $\Delta R = 0.598090 \times 10^6$ meters
d. 296.5 deg
CASE E. $\Delta V = 5.09664$ meters/sec
e. 299 deg
CASE F. $\Delta\Gamma = 0.798537 \times 10^{-1}$ deg
f. 354.5 deg

TIME OF FLIGHT 33.7 hr $C_{3E} = 4.6 \times 10^6 \left(\frac{\text{meters}}{\text{sec}}\right)^2$

CASE A. $\Delta Z = 0.201213 \times 10^6$ meters
a. 90 deg
CASE B. $\Delta\theta = 0.152333$ deg
b. 178 deg
CASE C. $\Delta \dot{Z} = 49.2961$ meters/sec
c. 4. 270 deg
CASE D. $\Delta V = 0.677462 \times 10$ meters/sec
d. 325 deg
CASE E. $\Delta R = 0.763812 \times 10^6$ meters
e. 328 deg
CASE F. $\Delta\Gamma = 0.806707 \times 10^{-1}$ deg
f. 357 deg

Fig. 3. Effect of Coordinate Error for Lunar Trajectory

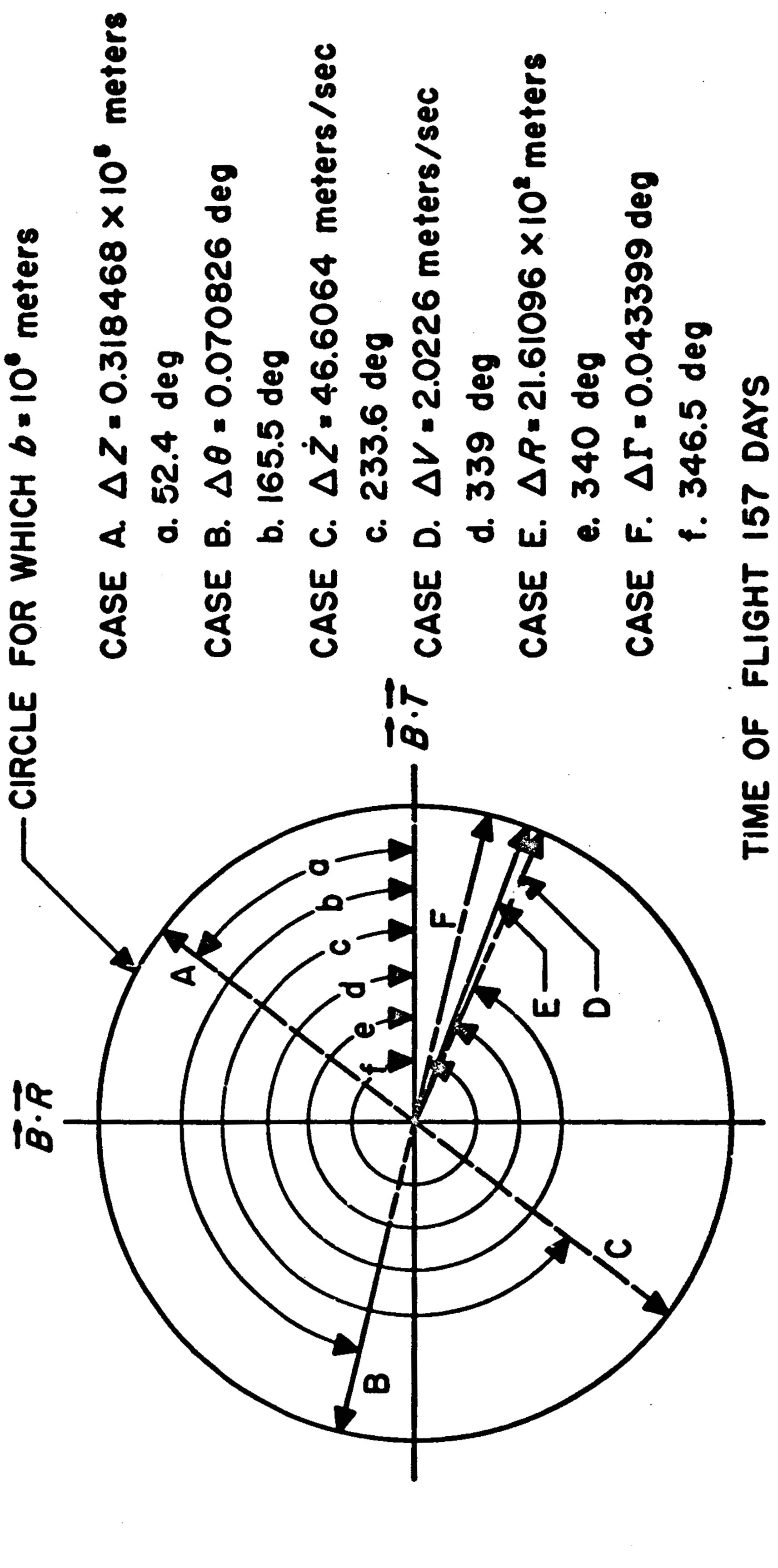


Fig. 4. Effect of Coordinate Error for Mars Trajectory

APPENDIX: THE CALCULATION OF ELEMENTS OF THE OSCULATING CONIC NEAR THE TARGET

The point at which the osculating conic near the target is computed will depend on whether the method of integration is direct (such as Cowell's method) or based on a variational scheme (for instance Encke's method). For a direct method the accuracy of the computation decreases when the path is near the attracting center, so that it may be advantageous to compute the elements of the conic when the vehicle has not yet reached the point of closest approach. For lunar trajectories it was found that a good point to compute the conic is when the probe is about three lunar diameters from the center of the moon. So that at this point or at the point of closest approach, whichever occurs first, the path angle Γ is computed. It is assumed that the rectangular coordinates $(X, Y, Z, \dot{X}, \dot{Y}, \dot{Z})$ of the relative motion of the vehicle to the center of mass are available.

$$\sin \Gamma = \frac{X\dot{X} + Y\dot{Y} + Z\dot{Z}}{RV} \tag{6}$$

where $-\frac{\pi}{2} \leq \Gamma \leq \frac{\pi}{2}$

The angular momentum,

$$C_1 = RV \cos \Gamma \tag{7}$$

The <u>vis viva</u>,

$$C_3 = V^2 - \frac{2GM}{R} \tag{8}$$

where G is the universal gravitational constant and M is the mass of the target.

The eccentricity,

$$e = \sqrt{\frac{1 + C_1^{\,2} C_3}{(GM)^2}} \tag{9}$$

The semilatus rectum,

$$p = \frac{C_1^{\,2}}{GM} \tag{10}$$

The polar angle or true anamoly

$$\theta = \cos^{-1}\left[\frac{1}{e}\left(\frac{p}{r} - 1\right)\right] \tag{11}$$

where $-\pi < \theta < \pi$

When $\Gamma < 0, \theta < 0$

$\Gamma > 0, \theta > 0$

$\Gamma = 0, \theta = 0$

The specification of the orientation of the conic can be done by means of unit vectors in the direction of the lower apsis, $\vec{\xi}$, the normal to the orbital plane $\vec{\zeta}$, and

$$\vec{\eta} = \vec{\zeta} \times \vec{\xi} \tag{12}$$

$$\vec{\zeta} = \frac{\vec{R} \times \vec{V}}{|R \times V|} \tag{13}$$

An auxiliary vector $\vec{M}$ is found by

$$\vec{M} = \vec{\zeta} \times \frac{\vec{R}}{R} \tag{14}$$

Then

$$\vec{\xi} = \frac{\vec{R}}{R} \cos \theta - \vec{M} \sin \theta \tag{15}$$

$$\vec{\eta} = \frac{\vec{R}}{R} \sin \theta + \vec{M} \cos \theta \tag{16}$$

The following formulas hold only if the conic is a hyperbola:

$$b = \frac{p}{\sqrt{e^2 - 1}} \tag{17}$$

The semimajor axis

$$a = \frac{GM}{C_3} \tag{18}$$

The unit vector in the direction of the asymptote is given by

$$\vec{S} = \frac{1}{e} \vec{\xi} + \sqrt{1 - \left(\frac{1}{e}\right)^2} \, \vec{\eta} \tag{19}$$

The impact parameter

$$\vec{B} = \left| \vec{B} \right| \left[\sqrt{1 - \left(\frac{1}{e}\right)^2} \, \vec{\xi} - \frac{1}{e} \, \vec{\eta} \right] \tag{20}$$

Also

$$T_x = \frac{S_y}{\sqrt{S_x{}^2 + S_y{}^2}} \tag{21}$$

$$T_y = - \frac{S_x}{\sqrt{S_x{}^2 + S_y{}^2}} \tag{22}$$

$$T_z = 0 \tag{23}$$

and

$$\vec{R} = \vec{S} \times \vec{T} \tag{1}$$

The reader may prefer to define $\vec{R}$ by

$$\vec{R} = \vec{T} \times \vec{S} \tag{1a}$$

instead, to make a positive $\vec{R}$ correspond to what is normally considered up with respect to the equatorial or ecliptic planes. Also a direct (as opposed to a retrograde) path corresponds to a positive $\vec{T}$.